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D. Chinhyung Cho, E.L. Johnson, M.W. Padberg, M.R. Rao

IRIA

CENTRE DE ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél.: 954 90 20

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by

D. CHINHYUNG CHO*)

Ellis L. JOHNSON**)

Manfred W. PADBERG***)

M.R. RAO****)

*) GBA, New York University, New York, N.Y., U.S.A.

**) IBM Research, Yorktown Heights, N.Y., U.S.A. and Universität Bonn, Bonn, R.F.A.

***) GBA, New York University, New York, N.Y., U.S.A. and INRIA, Rocquencourt, France.

****) Indian Institute of Management, Bangalore, India.

Résumé :

Nous considérons un problème standard de la recherche opérationnelle concernant l'implantation des installations appelé "uncapacitated plant location problem" (PLP) dans la littérature anglaise. Nous voyons ce problème ici comme un problème défini sur un graphe où il faut trouver un ensemble de sommets indépendants. Pour le problème PLP, nous montrons plusieurs inégalités valides dont quelques unes définissent des facettes du polytope avec des sommets zéro ou un associé à ce problème. Nous déduisons des conditions nécessaires et suffisantes pour les facettes dites triviales et, en plus, des conditions nécessaires pour des facettes générales. Pour le cas particulier de ce problème à trois installations et à trois destinations ou plus, nous donnons une description complète du polytope associé à des inégalités linéaires.

Abstract :

The uncapacitated plant location problem is considered as a node-packing problem. For this problem, several valid inequalities and facets are discussed. Necessary and sufficient conditions for trivial facets along with necessary conditions for non-trivial facets are derived. In addition, all of the facets for the case of 3 plants and 3 or more destinations are identified.

1. Introduction.

The plant location problem is the problem of finding the optimal location of plants so as to minimize a cost function while satisfying given demands. The problem is called the uncapacitated (or simple) plant location problem if each plant under consideration has the ability to satisfy the total demand. The plant location problem is a standard problem of the OR/MS literature and is of interest not only for its own merits but also for its close similarities to other classes of real-world problems. (See [10] for a detailed bibliography.)

Weber [24], who may be credited with the formulation of the problem, and other researchers attempted to solve the problem by way of locating the centeroids. This approach often resulted in locating the plants at impossible locations; for example, at the top of a mountain or in the middle of a lake. This impossibility of the optimal location was later resolved by constraining the problem to allow only a finite number of feasible locations.

In fact, since 1954 when Hirsch and Dantzig [17] introduced a linear programming method for solving the fixed charge problem, most of the studies have been carried out along a linear programming approach using 0-1 decision variables. The majority of the researchers attempted to solve this mixed integer programming problem by using the branch and bound technique, see e.g. Efroymson and Ray [9] and Spielberg [25]. However, the computational results have not been satisfactory, especially when faced with problems of a large size. This can be expected because the problem is NP-complete, i.e. it belongs to the class of hard combinational optimization problems which to date cannot be solved by polynomially bounded algorithms [12].

Recently, several researchers have begun to study facets of the uncapacitated plant location problem. (See Guignard [16] and Cornuejols and Thizy [6].) This approach deserves attention since facets are "stongest cutting planes". One can thus reasonably expect to improve computational results substantially even if one can identify only a subset of these facets. This point has been well illustrated recently with related work on the travelling salesman problem by Grötschel and Padberg [14, 15], Crowder and Padberg [7] and Padberg and Hong [23].

Our objective, then, is to investigate some of these facet inequalities that describe the integer polyhedron of the uncapacitated plant location problem.

In section 2, we formulate the problem as the node-packing problem (PLP) and introduce some notation that will be used through-out the study. Once formulated in this way, it follows that the constraint matrix of (PLP) is totally unimodular if the number of plants, p, is less than 3 or the number of destinations, d, is less than 3. This implies that the linear programming relaxation of (PLP) will give an integer solution and thus it is of no interest to us here. However, if $p \geq 3$ and $d \geq 3$, then linear programming solution will not always be integer and thus our concern is how we can eliminate these non-integer solutions.

Section 3 deals with a particular class of valid inequalitites which remove some fractional extreme points of the polyhedron associated with the linear programming relaxation of (PLP). It is demonstrated that all of the valid inequalities studied to date are members of this class.

In section 4, we show that the non-negativity constraints and clique constraints are all facets and prove several necessary and sufficient conditions that characterize facets.

The results of section 4 and the necessary conditions for non-trivial facets of section 5 enable us to give a description of all facets of (PLP) when $p = 3$ and $d \geq 3$. In fact, theorem 6.1 states that all of these facets correspond to odd-cycles of length 9 in the associated intersection graph.

In a sequel to this paper [4], we discuss several lifting procedures and present necessary and sufficient conditions for non-trivial facets with 0-1 coefficients. We also derive a family of facets and describe all facets for the case of several plants $(p \geq 3)$ and three destinations.

2. Problem formulation and notation.

The uncapacitated plant location problem arises from the following situation. Suppose that, at p possible locations, plants can be built necessitating a fixed investment f_i at location i ($i = 1,2,\ldots,p$) for producing a homogeneous good demanded from destination j ($j = 1,2,\ldots,d$). Assuming that the capacity of each plant is sufficiently large to satisfy the total demand, one has to decide at which locations to build the plants so as to minimize the total cost of satisfying the destination's demands.

Denoting by x_{ij} the fraction of destination j's demand filled by a plant i and letting $y_i' = 1$ if a plant is located in location i ($y_i' = 0$ if not), the problem can be formulated as the following mixed integer programming problem.

$$\text{Min.} \quad \sum_j \sum_i c_{ij} x_{ij} \quad + \quad \sum_i f_i y_i'$$

subject to

$$\sum_i x_{ij} = 1 \qquad \text{for } j = 1,2,\ldots,d$$

$$0 \le x_{ij} \le y_i' \qquad \text{for } i = 1,2,\ldots,p;\ j = 1,2,\ldots,d$$

$$y_i' = 0 \text{ or } 1 \qquad \text{for } i = 1,2,\ldots,p$$

Here, c_{ij} are the transportation costs involved in satisfying the total demand of destination j from plant i and, moreover, transportation costs are assumed to be linear.

This formulation can be transformed into a set-packing problem by interchanging the variables y_i' with $1-y_i$ and by converting the set of d equality constraints into less-than-or-equal-to constraints. The latter can be accomplished by introducing an artificial variable for each equation having a "large enough" cost (e.g. $\theta_j > \sum_i c_{ij}$) and subsequent elimination of these variables. More precisely, the interchange of variables y_i' with $1-y_i$ transforms the problem into the form

$$\text{Min.} \sum_j \sum_i c_{ij} x_{ij} - \sum_i f_i y_i + \sum_i f_i$$

subject to

$$\text{(PLPE)} \quad \begin{array}{ll} \sum_i x_{ij} = 1 & \text{for } j = 1,2,\ldots,d \\ x_{ij} + y_i \leq 1 & \text{for } i = 1,2,\ldots,p;\ j = 1,2,\ldots,d \\ y_i = 0 \text{ or } 1 & \text{for } i = 1,2,\ldots,p \\ x_{ij} \geq 0 & \text{for } i = 1,2,\ldots,p;\ j = 1,2,\ldots,d. \end{array}$$

Applying the second transformation, (PLPE) can be brought to the following equivalent form:

$$\text{Max.} \sum_j \sum_i (\theta_j - c_{ij}) x_{ij} + \sum_i f_i y_i - \sum_i f_i - \sum_j \theta_j$$

subject to

$$\text{(PLP)} \quad \begin{array}{lll} \sum_i x_{ij} \leq 1 & \text{for } j = 1,2,\ldots,d & (2.1) \\ x_{ij} + y_i \leq 1 & \text{for } i = 1,2,\ldots,p;\ j = 1,2,\ldots,d & (2.2) \\ y_i = 0 \text{ or } 1 & \text{for } i = 1,2,\ldots,p & \\ x_{ij} \geq 0 & \text{for } i = 1,2,\ldots,p;\ j = 1,2,\ldots,d. & \end{array}$$

This formulation is the required set-packing version of the plant location problem.

Remark 2.1. (PLPE) differs from (PLP) by the fact that the former must satisfy the constraints (2.1) with equalities.

Remark 2.2. In this formulation, $y_i = 0$ means that a plant is located at i and $y_i = 1$ means the opposite case. Also, even though x_{ij} is allowed to take a fractional value, there exists an optimal solution with $x_{ij} = 0$ or 1 if all variables y_i are fixed at 0 or 1.

Let A be the matrix whose elements are the coefficients of the constraints (2.1) and (2.2) of (PLP). Note that A is a (pd+p) x (pd+p) matrix whose elements are either 0 or 1. The intersection graph $G = (N,E)$ is constructed by assigning a node for each column a^i of A and an undirected edge for each pair of nodes whose corresponding columns are non-orthogonal, i.e. $|N| = pd+p$ and there is an edge between the node i and the node j if and only if $a^i.a^j \neq 0$. It follows that (PLP) is equivalent to the node-packing prolbem (NP) in this intersection graph in the sense that they both have the same solution set (including the set of optimal solutions) if for each node i the node weight is defined to be the cost associated with column i. (see [8,13, 19,21].)

For convenience, we define some notation.

1) $P = \{1,2,\ldots,p\}$ with $|P| = p$ and $D = \{1,2,\ldots,d\}$ with $|D| = d$.

2) $L^{pd} = \{(x,y) \in R^{p(d+1)} \mid \sum_i x_{ij} \leq 1 \ \forall\ j \in D;\ x_{ij} + y_i \leq 1,\ x_{ij} \geq 0,\ y_i \geq 0\ \forall\ i \in P,\ j \in D\}$

This is the polyhedron associated with the linear programming relaxation of (PLP) with p plants and d destinations. Note that $\dim(L^{pd}) = p(d+1)$.

3) $LE^{pd} = \{(x,y) \in R^{p(d+1)} \mid \sum_i x_{ij} = 1 \ \forall\ j \in D;\ x_{ij} + y_i \leq 1,\ x_{ij} \geq 0,\ y_i \geq 0\ \forall\ i \in P,\ j \in D\}$

This is the polyhedron associated with the linear programming relaxation of (PLPE) with p plants and d destinations.

4) $L_I^{pd} = \text{conv}\{L^{pd} \cap Z^{p(d+1)}\}$ where conv means the convex hull and $Z^{p(d+1)}$ is the set of all 0-1 vectors of deminsion $p(d+1)$. This is the integer polyhedron associated with (PLP), i.e. the convexified solution space of (PLP).

5) $Q^{pd} = \{(\pi,\mu)\in R^{p(d+1)} \mid \sum_j \pi_{\sigma(j)j} + \sum_{i\in S}\mu_i \le 1 \ \forall\ S \subseteq P;\ \pi_{ij} \ge 0,\ \mu_i \ge 0\ \forall\ i\in P,\ j\in D\}$

where σ is an onto function from D into $\bar{S}$ (= P-S). This is the anti-blocker [11] of L_I^{pd} and thus, if $\pi x + \mu y \le 1$ is a facet of L_I^{pd}, then (π,μ) is a vertex of Q^{pd}.

6) $F^{pd} = \{(\pi,\mu)\in R^{p(d+1)} \mid \pi x + \mu y \le \pi_0$ is a facet of L_I^{pd} for some $\pi_0\}$

This is the set consisting of all the possible facets.

7) $F_t^{pd} = \{(\pi,\mu)\in R^{p(d+1)} \mid \pi x + \mu y \le \pi_0$ where $\pi_0 = 0$ or 1 and $\pi x + \mu y$ is equal to $\sum_i x_{ij}$ or $x_{ij} + y_i$ or $-x_{ij}$ or $-y_i$ for some $i\in P$, $j\in D$ $\}$

This is the set consisting of all clique constraints and non-negativity constraints.

8) $H_{\pi,\mu} = L_I^{pd} \cap \{(x,y)\in R^{p(d+1)} \mid \pi x + \mu y = 1\}$, where $\pi x + \mu y \le 1$ is a valid inequality for L_I^{pd}. This is the set consisting of all the feasible points that lie on the plane $\pi x + \mu y = 1$.

9) $I_j = \{i\in P \mid \pi_{ij} > 0\}$ for $j = 1,2,\ldots,d$ for a given (π,μ) and, likewise, $J_i = \{j\in D \mid \pi_{ij} > 0\}$ for $i = 1,2,\ldots,p$.

10) Let $(\pi,\mu) \in Q^{pd}$. Then, we partition π as $\pi = (\pi^1,\pi^2,\ldots,\pi^d)$ where $\pi^j = (\pi_{1j},\pi_{2j},\ldots,\pi_{pj})$ for $j = 1,2,\ldots,d$ and accordingly for x as $x = (x^1,x^2,\ldots,x^d)$.

11) We let $G = (N, E)$ be the intersection graph associated with (PLP). Recall that each node in N corresponds to a column of A whose elements are the coefficients of the constraints (2.1) and (2.2). Hence, we denote the nodes accordingly by their corresponding variables of the constraints (2.1) and (2.2). In other words, N consists of all the nodes y_i, $i = 1,2,\ldots,p$ and x_{ij}, $i = 1,2,\ldots,p$; $j = 1,2,\ldots,d$ while E contains all the edges (x_{ij},y_i), $i = 1,2,\ldots,p$; $j = 1,2,\ldots,d$ and $(x_{\mu j},x_{\nu j})$, $\mu \neq \nu$; $j = 1,2,\ldots,d$.

12) Let $G^S = (N^S, E^S)$ be a subgraph of G induced by N^S where $N^S \subseteq N$, i.e. G^S has the node set N^S and all edges of G whose both end points are in N^S. Let $I^S = \{i \in P \mid y_i \in N^S\}$ and $J^S = \{j \in D \mid x_{ij} \in N^S$ for some $i \in P\}$. Also, let S be an $|I^S| \times |J^S|$ matrix whose elements are such that $s_{ij} = 1$ if $(y_i,x_{ij}) \in E^S$ and $s_{ij} = 0$ if $(y_i,x_{ij}) \notin E^S$. Note that $s_{ij} = 1$ implies that a plant i can cover a destination j. We say that S is an adjacency matrix if there is no zero column, i.e. every destination in J^S can be covered by some plant $i \in I^S$, and no zero row, i.e. each plant $i \in I^S$ can cover at least one destination $j \in J^S$. Vice versa, any $|I^S| \times |J^S|$ zero-one matrix $S = (s_{ij})$ with $I^S \subseteq P$ and $J^S \subseteq D$ having no zero column and no zero row is an adjacency matrix and there exists a corresponding subgraph $G^S = (N^S,E^S)$ of G, where $N^S = \{x_{ij} \in N \mid i \in I^S,\ j \in J^S \text{ and } s_{ij} \neq 0\} \cup \{y_i \in N \mid i \in I^S\}$ and $E^S = \{(x_{\mu j},x_{\nu j}) \in E \mid \mu \neq \nu,\ \mu \in I^S,\ \nu \in I^S,\ j \in J^S \text{ and } s_{\mu j}\cdot s_{\nu j} \neq 0\} \cup \{(y_i,x_{ij}) \in E \mid i \in I^S,\ j \in J^S \text{ and } s_{ij} \neq 0\}$.

13) For any subgraph G^S of G, we let $\alpha(G^S)$ be the independence number of G^S, i.e. maximum node packing for the intersection graph G^S, and we let $\beta(G^S)$ be the covering number of G^S, i.e. the minimum number of plants $i \in I^S$ necessary to

cover all destinations $j \in J^S$. If no ambiguity arises, we simply write α or β. We call S to be a <u>maximal adjacency matrix</u> if changing the zero element to one decreases $\beta(G^S)$ by one or increases $\alpha(G^S)$ by one. We call an edge of E^S to be <u>critical</u> if its removal increases $\alpha(G^S)$.

14) For an adjacency matrix S, let $L_I^S = L_I^{pd} \cap \{(x,y) \in R^{p(d+1)} \mid x_{ij} = 0 \text{ for } s_{ij} = 0,$ and $y_i = 0,\ x_{ij} = 0 \text{ for } i \notin I^S,\ j \notin J^S\}$.

15) An adjacency matrix S is called a <u>pd-adjacency matrix</u> if i) G^S is connected, ii) there exist at least one zero element in each column, and iii) $|I^S| \geq 3$ and $|J^S| \geq 3$. The corresponding subgraph G^S is called a <u>pd-subgraph</u>.

16) $\lceil a \rceil$ is the smallest integer greater than or equal to a.

3. Valid inequalities.

An inequality $ax + by \le a_0$ is called a valid inequality with respect to L_I^{pd} if and only if $L_I^{pd} \subseteq \{(x,y) \in R^{p(d+1)} \mid ax + by \le a_0\}$. The valid inequalities (or cuts) are very useful in removing the fractional extreme verticies that result from the linear relaxation of the integer programming problems. A facet may be viewed as the 'strongest' valid inequality in a sense that it is a valid inequality that intersects the integer polyhedron in a face of the highest possible dimension. However, it should be noted that intersecting L^{pd} by a valid inequality may introduce new fractional extreme points that in turn have to be removed again.

By noting the correspondence between a pd-subgraph $G^S = (N^S, E^S)$ and its $|I^S| \times |J^S|$ adjacency matrix S, we get the following theorem.

Theorem 3.1. For any pd-subgraph $G^S = (N^S, E^S)$, the inequality

$$\sum_{i \in I^S} \sum_{j \in J^S} s_{ij} x_{ij} + \sum_{i \in I^S} y_i \le |I^S| + |J^S| - k \qquad (3.1)$$

is a valid inequality of L_I^{pd} if and only if k is an integer satisfying $k \le \beta(G^S)$. Furthermore, $\alpha(G^S) = |I^S| + |J^S| - \beta(G^S)$.

Proof:

($\Rightarrow$)Suppose (3.1) is a valid inequality and suppose $k > \beta(G^S) = \beta$. Since β plants are sufficient to cover all destinations $j \in J^S$, the left hand side of (3.1) can become $|J^S| + |I^S| - \beta$ which is greater than the right hand side of (3.1). Hence, we have a contradiction.

($\Leftarrow$)Consider any feasible solution such that η plants from I^S are open.

Case 1. $\eta \geq \beta$. Then, $y_i = 1$ for exactly $|I^S|-\eta$ plants. But, $|I^S|-\eta \leq |I^S|-\beta \leq |I^S|-k$. Thus, even if all $|J^S|$ destinations are covered, the left hand side of (3.1) will be less than or equal to $|I^S|+|J^S|-k$.

Case 2. $\eta < \beta$. Since β plants are required to cover all $|J^S|$ destinations, η plants can cover at most $|J^S|-(\beta-\eta)$ destinations. Therefore, the left hand side of (3.1) can be at most $|J^S|-(\beta-\eta)+|I^S|-\eta$ which is less than or equal to $|I^S|+|J^S|-k$.

Now, we prove that $\alpha(G^S) = |I^S|+|J^S|-\beta$. First, we note that $\alpha(G^S) \leq |I^S|+|J^S|-\beta$ because (3.1) holds when $k = \beta$. Without loss of generality, assume that the β plants that can cover all destinations are numbered 1 to β, i.e. $y_i = 0$ for $i = 1,2,\ldots,\beta$ and $y_i = 1$ for the remaining plants in I^S. Since the β plants can cover all destinations, for each $j \in J^S$ we can set $x_{ij} = 1$ where i is such that (x_{ij},y_i) is an edge of G^S and $1 \leq i \leq \beta$. Hence, $\alpha(G^S) \geq |I^S|+|J^S|-\beta$. Therefore, $\alpha(G^S) = |I^S|+|J^S|-\beta$. Δ

Remark 3.1. If (3.1) is a facet, then $k = \beta(G^S)$ and the right hand side becomes $\alpha(G^S)$.

A valid inequality is useful only when it can remove some fractional extreme points. Cornuejols, Fisher and Nemhauser (C.F.N) showed the following characterization of an extreme point for the LP relaxation of (PLPE). (See [5].)

Theorem 3.2. For a given non-integer solution (x^f,y^f) contained in LE^{pd}, let $P_f = \{i\in P|\ 0 < y_i^f < 1\}$, $D_f = \{j\in D|\ x_{ij}^f$ non-integer for some i, and $x_{ij}^f = 0$ or $1-y_i^f$ for all $i \in P_f\}$ and F be the $|P_f| \times |D_f|$ matrix whose elements are $f_{ij} = \begin{cases} 1 & \text{if } x_{ij}^f > 0 \\ 0 & \text{if } x_{ij}^f = 0 \end{cases}$. Then, the non-integer solution (x^f,y^f) is an extreme

<u>point of LE^{pd} if and only if</u>

<u>1) $y_i^f = 1 - \max_j x_{ij}^f$ for all $i \in P_f$</u> ,

<u>2) for each $j \in D$, there exists at most one $i(j) \in P$ with $0 < x_{i(j)j}^f \leq 1 - y_{i(j)}^f$</u> ,

<u>3) rank of F equals $|P_f|$</u>.

<u>Proof</u>: See [5, theorem 6].

If we further define $P_0 = \{i \in P \mid y_i^f = 0\}$, $P_1 = \{i \in P \mid y_i^f = 1\}$, $D_1 = \{j \in D \mid x_{ij}^f$ integer for all $i \in P\}$, $D_2 = \{j \in D \mid 0 < x_{i(j)j}^f < 1 - y_{i(j)}^f$ for precisely one $i(j) \in P\}$, then it is easy to see that $\{P_0, P_1, P_f\}$ and $\{D_1, D_2, D_f\}$ partition P and D, respectively. In fact, any fractional extreme point must have the following x_{ij}'s.

		destinations		
		D_1	D_f	D_2
plants	P_1	0	0	0
	P_f	0	0 or fractionals **	0 or fractionals
	P_0	0 or* 1	0	0 or fractionals

*: Each column has exactly one 1.

**: There are no zero-rows and each column has at least two fractional numbers. This $|P_f| \times |D_f|$ matrix corresponds to F.

If the rank of F equals $|P_f|$, F contains a $|P_f| \times |D_f'|$ non-singular submatrix B where $|P_f| = |D_f'|$ and $D_f' \subseteq D_f$, and the fractional parts of the extreme point (x^f, y^f) can be fully identified by solving $z^t B = e^t$ where z^t and e^t are the transpose of z_i's and one's, respectively. This is so because an extreme point of LE^{pd} must satisfy $\sum_i x_{ij} = 1$, $x_{ij} + y_i \leq 1$ and the three conditions above in theorem 3.2. In other words, we let

$$y_i^f = 1 - z_i \qquad \text{for all } i \in P_f$$

$$x_{ij}^f = \begin{cases} z_i & \text{if } f_{ij} = 1 \\ 0 & \text{if } f_{ij} = 0 \end{cases} \qquad \forall\, i \in P_f,\ j \in D_f \text{ or } \forall\, i \in P_f - i(j),\ j \in D_2$$

$$x_{i(j)j}^f = 1 - \sum_{i \in P_f - i(j)} x_{ij}^f \qquad \forall\, j \in D_2$$

Note that B is an adjacency matrix.

Vice versa, it can be seen that there corresponds an extreme point of LE^{pd} for any non-singular $k \times k$ adjacency matrix B if $k \le \min\{p,d\}$ and $z^t B = e^t$ has a non-negative solution. Furthermore, an extreme point of LE^{pd} is also an extreme point of L^{pd}. Hence, we can derive a valid inequality of L_I^{pd} that removes some extreme points of L^{pd} as follows.

Theorem 3.3. Let B be an $|I^B| \times |J^B|$ non-singular adjacency matrix where $|I^B| = |J^B|$ and for which $z^t B = e^t$ has non-negative z_i's. If $\beta(G^B) > \sum_{i \in I^B} z_i$, then the inequality

$$\sum_{i \in I^B} \sum_{j \in J^B} b_{ij} x_{ij} + \sum_{i \in I^B} y_i \le |I^B| + |J^B| - \beta(G^B) \qquad (3.2)$$

is a valid inequality of L_I^{pd} and removes some fractional extreme points of L^{pd}.

Proof:

From theorem 3.1, (3.2) is a valid inequality of L_I^{pd}. Furthermore, from theorem 3.2, there exists a fractional point satisfying

$$\sum_{i \in I^B} \sum_{j \in J^B} b_{ij} x_{ij} + \sum_{i \in I^B} y_i = |J^B| + |I^B| - \sum_{i \in I^B} z_i$$

Therefore, the valid inequality (3.2) removes some fractional extreme points because $\beta(G^B) > \sum_{i \in I^B} z_i$. Δ

We now consider some special cases of (3.2). Let C be a k x k cyclic matrix whose rows are 0-1 vectors in which t consecutive ones are successively moved one position to the right. Furthermore, assume k and t are relatively prime with $t < k$. Then, C is a non-singular adjacency matrix and $\beta(G^C) = \left\lceil \frac{k}{t} \right\rceil > \sum_{i \in I^C} z_i = \frac{k}{t}$. Hence, the valid inequality (3.2) becomes

$$\sum_{i \in I^C} \sum_{j \in J^C} c_{ij} x_{ij} + \sum_{i \in I^C} y_i \le 2k - \left\lceil \frac{k}{t} \right\rceil \tag{3.3}$$

In particular, if there are k-1 one's per row, i.e. $t = k-1$, then $\beta(G^C) = 2$ and the valid inequality (3.2) becomes

$$\sum_{i \in I^C} \sum_{j \in J^C} c_{ij} x_{ij} + \sum_{i \in I^C} y_i \le 2k - 2 \tag{3.4}$$

On the other hand, if k is odd and $t = 2$, then C is non-singular with $\beta(G^C) = \left\lceil \frac{k}{2} \right\rceil$ and the valid inequality (3.2) becomes

$$\sum_{i \in I^C} \sum_{j \in J^C} c_{ij} x_{ij} + \sum_{i \in I^C} y_i \le 2k - \left\lceil \frac{k}{2} \right\rceil \tag{3.5}$$

We note that the inequality (3.3) is exactly the same as the one obtained in [5], (3.4) is the same as the one obtained in [16] and (3.5) is the odd-cycle inequality obtained in [21]. We shall show, in a sequel to this paper [4], that one can lift the inequality (3.3) and (3.5) to be a facet of L_I^{pd} whereas in [16] it has been shown that (3.4) defines a facet of L_I^{pd}. (See also [6].) Note that (3.3) remains a valid inequality for L_I^{pd} if the assumption that k and t are relatively prime is dropped. This is an immediate consequence of Theorem 3.1. In the sequel [4] we show how these valid inequalities can be extended to be facets of the polyhedron L_I^{pd}.

4. Trivial facets.

An inequality $\pi t \le \pi_0$ is called a facet of a polyhedron P if i) $\pi t \le \pi_0$ for all $t \in P$, and ii) there exists n affinely independent points $t^i \in P$ such that $\pi t^i = \pi_0$ for $i = 1,2,\ldots,n$ where dim (P) = n. The first property implies that $\pi t \le \pi_0$ is a valid inequality for P and the second property implies that there are n linear independent vectors on the face of the polyhedron if $\pi_0 \ne 0$.

From this definition of a facet, we immediately get the following theorem about the non-negativity constraints. (See Balas and Padberg [1] for a similar argument.)

Theorem 4.1. Let $\pi x + \mu y \le \pi_0$ be a facet of L_I^{pd}. Then, $\pi_0 = 0$ if and only if $\pi x + \mu y = -x_{ij}$ or $\pi x + \mu y = -y_i$.

Proof:

For notational convenience, we let $t_{(i-1)d+j} = x_{ij}$ for $i = 1,2,\ldots,p$; $j = 1,2,\ldots,d$ and $t_{pd+i} = y_i$ for $i = 1,2,\ldots,p$. Then, the above statement can be written as follows: Let $\pi t \le \pi_0$ be a facet of L_I^{pd}. Then $\pi_0 = 0$ if and only if $\pi t = -t_i$.

($\Rightarrow$)Suppose $\pi_0 = 0$. Then, since any unit vector is feasible, we must have $\pi_i \le 0$ for all $i = 1,2,\ldots,pd+p$. Let $N^- = \{i \mid \pi_i < 0\}$. Then, there are $pd+p-|N^-|$ linearly independent vectors satisfying $\pi t = 0$. However, for $\pi t \le 0$ to be a facet, we must have $pd+p-1$ linearly independent vectors satisfying $\pi t = 0$. Therefore, $|N^-| = 1$.

($\Leftarrow$)Suppsoe $\pi_0 \ne 0$, but $\pi t = -t_i$. Clearly, $\pi_0 < 0$ is not possible because 0 vector is a feasible solution. If $\pi_0 > 0$, then there exists no feasible vector satisfying $\pi t = \pi_0$ contradicting the fact that $\pi t \le \pi_0$ is a facet. Therefore, $\pi_0 = 0$. Δ

<u>Remark 4.1</u>. If $\pi_0 \neq 0$, we can assume without loss of generality that $\pi_i \geq 0$ for $i = 1,2,\ldots,pd+p$ and $\pi_0 = 1$. This can be seen as follows. Suppose $N^- \neq \emptyset$. Let $i \in N^-$. Then, since $\pi_0 \neq 0$, there must exist $\bar{t} \in L_I^{pd}$ such that $\bar{t}_i = 1$ and $\pi\bar{t} = \pi_0$. (Otherwise, if $\bar{t}_i = 0$ for all solutions such that $\pi\bar{t} = \pi_0$, then all such solutions also satisfy the facet $-t_i \le 0$ with equality. Consequently, $\pi t \le \pi_0$ can not be a facet.) But, if we define t^* to be $t_j^* = \bar{t}_j$ for $j \neq i$ and $t_i^* = 0$ for $i \in N^-$, then from the non-negativity of A, we have $At^* \le A\bar{t} \le e$, i.e. t^* feasible, and $\pi t^* = \pi\bar{t} + |\pi_i| > \pi_0$ contradicting the fact that $\pi t \le \pi_0$ is a facet. Consequently, $N^- = \emptyset$ and we must have $\pi_i \geq 0$ for $i = 1,2,\ldots,pd+p$. Furthermore, $\pi_0 > 0$, which enables us to divide both sides of inequality by π_0.

We, now, give necessary and sufficient conditions for the facets to be clique constraints. Clique constraints have been shown to be facets by Padberg [21] and we will use this fact in the proof of the next theorem repeatedly.

<u>Theorem 4.2</u>. <u>Let $\pi x + \mu y \le 1$ be a facet of L_I^{pd}. Then, the following statements are true.</u>

1) <u>$\pi x + \mu y = x_{ij} + y_i$ for some $i \in P$, $j \in D$ if and only if $\sum_i \mu_i = 1$.</u>

2) <u>$\pi x + \mu y = x_{ij} + y_i$ if and only if there exists j such that $\pi^j \neq 0$ and exactly one $\pi_{ij} > 0$.</u>

3) <u>$\pi x + \mu y = \sum_i x_{ij}$ for some $j \in D$ if and only if $\pi^j > 0$.</u>

<u>Proof</u>:

1)($\Rightarrow$)This is obvious because we have $\mu_i = 1$, and $\mu_k = 0$ for $k \neq i$.

($\Leftarrow$)If $(\pi,\mu) \in F^{pd}$, then $\sum_j \pi_{\sigma(j)j} + \sum_{i\in S} \mu_i \le 1$ must hold for all $S \subseteq P$ where σ is an onto function from D into $\bar{S}$. Take $\bar{S} = \{i\}$. Then, we must have $\sum_i \pi_{it} + \sum_{\substack{k\\k\neq i}} \mu_k \le 1$ because it is feasible for a plant i to supply all destinations. This is equivalent to $\sum_t \pi_{it} \le \mu_i$ because of the condition $\sum_k \mu_k = 1$. Suppose $\pi_{ij}>0$, but $\pi x + \mu y \neq x_{ij} + y_i$. Then, all solutions $(x,y)\in H_{\pi,\mu}$ must have either $y_i = 1$ or $y_i = 0$ and $x_{ij} = 1$. This is so because, if there exist a solution $(x^*,y^*) \in H_{\pi,\mu}$ such that $y_i^* = 0$ and $x_{ij}^* = 0$, then we can let $(\bar{x},\bar{y})$ be same with (x^*,y^*) except $\bar{y}_i = 1$ and $\bar{x}_{it} = 0$ for $t \in J_i$ so that $(\bar{x},\bar{y})$ is feasible but $\pi\bar{x} + \mu\bar{y} = \pi x^* + \mu y^* + \mu_i - \sum_{t\in J_i} \pi_{it} + \pi_{ij} > 1$ leading to the contradiction.

2)($\Rightarrow$)Obvious.

($\Leftarrow$)Without loss of generality, suppose $\pi^j \neq 0$ and $\pi_{1j} > 0$, $\pi_{kj} = 0$ for $k \ge 2$. Furthermore, suppose $\pi x + \mu y \neq x_{1j} + y_1$. Now, there must exist an integer vector $(x,y) \in H_{\pi,\mu}$ such that $x_{1j} + y_1 < 1$, which implies $x_{1j} = 0 = y_1$. This is so because $x_{1j} + y_1 \le 1$ is a facet and (x,y) are integers. Since π_{1j} is the only positive component (i.e. $\pi_{qj} = 0$ if $q \neq 1$), if we define a new vector $(\bar{x},\bar{y})$ to be $\bar{x}_{1j} = 1$, $\bar{x}_{qj} = 0$ if $x_{qj} = 1$, and all other components being same with (x,y), th[illegible] $(\bar{x},\bar{y}) \in L_I^{pd}$ and $\pi\bar{x} + \mu\bar{y} = \pi x + \mu y + \pi_{1j} > 1$. This is a contradiction to the fa[illegible] $\pi x + \mu y \le 1$ is a facet.

3)($\Rightarrow$)This is obvious because $\pi^j = (1,1,\ldots,1)$.

($\Leftarrow$)Suppose $\pi^j > 0$ but $\pi x + \mu y \neq \sum_i x_{ij}$. Then, we must have an integer vector $(x,y) \in H_{\pi,\mu}$ such that $\sum_i x_{ij} < 1$ which implies $x^j = 0$. This is so because $\sum_i x_{ij} \le 1$ defines a facet and (x,y) are integers. Consequently, $y_i = 1$ holds for all $i = 1,2,\ldots,p$. Otherwise, if $y_k = 0$ for some k, then we can set $x_{kj} = 1$ while maintaining feasibility for L_I^{pd} but, then, $\pi x + \mu y + \pi_{kj} > 1$ which contradicts the assumption that $\pi x + \mu y \leq 1$ is a facet. Hence, $\sum_i \mu_i = 1$ holds and by 1) we can not have $\pi^j > 0$. This is a contradiction.

We have thus shown that all inequalities of the set F_t^{pd} defined under point 7) of section 2 define facets of L_I^{pd}. We will call these facets <u>trivial</u> facets of L_I^{pd} .

5. Non-trivial facets.

In this section, we will discuss some necessary conditions for non-trivial facets. By remark 4.1, we can restrict ourselves to non-negative (π,μ) when we consider nontrivial facets. Furthermore, the right hand side of the related inequality, π_o, is understood to equal 1 in the following discussion.

Theorem 5.1. Let $(\pi,\mu) \in F^{pd} - F_t^{pd}$. Then, the following statements are true.

1) If $\pi_{ij} > 0$ for some j, then $\mu_i \geq \pi_{ij}$ for all i.

2) If $\mu_i > 0$, then there exist $j \neq k$ such that $\pi_{ij} > 0$ and $\pi_{ik} > 0$, i.e. $|J_i| \geq 2$.

3) If $I_j \neq \emptyset$, then $|I_j| \geq 2$ and there exist $i \neq k$ such that

$\max\{\pi_{qj} \mid q \in I_j\} = \pi_{ij} = \pi_{kj}$.

4) If $I_j \neq \emptyset$ and $I_k \neq \emptyset$ for $j \neq k$, then $I_j \neq I_k$, i.e. I_j and I_k must be distinct.

Proof:

1) Since (π,μ) is not a clique constraint, there exists an integer vector $(x,y) \in H_{\pi,\mu}$ such that $x^j = 0$ for some $j \in D$. Otherwise, $\sum_i x_{ij} = 1$ for all $j = 1,2,\ldots,d$ which would be clique constraints. Consequently, without loss of generality, we can let $y_i = 1$ for all $i \in I_j$. Because, if not, then $y_k = 0$ for some $k \in I_j$ and so by defining a new vector $(\bar{x},\bar{y})$ to be $\bar{x}_{kj} = 1$ for some $\pi_{kj} > 0$ others being same with (x,y) we get $(\bar{x},\bar{y}) \in L_I^{pd}$ and $\pi\bar{x} + \mu\bar{y} = \pi x + \mu y + \pi_{kj} > 1$ contradicting the fact that $\pi x + \mu y \leq 1$ is a facet.

Suppose, now, that $\mu_q < \pi_{qj}$ for some $q \in I_j$. Then, by defining a new vector (x^*,y^*) to be $x^*_{qj} = 1$, $y^*_q = 0$ others being same with (x,y), we get $(x^*,y^*) \in L_I^{pd}$ and $\pi x^* + \mu y^* = \pi x + \mu y - \mu_q + \pi_{qj} > 1$. This is a contradiction.

2) Since (π,μ) is not a clique constraint, there exists an integer vector

$(x,y) \in H_{\pi,\mu}$ such that $x_{ij} = 0$ and $y_i = 0$ for i, j such that $\pi_{ij} > 0$ and $\mu_i > 0$. Otherwise, for such i and j, $x_{ij} + y_i = 1$, which would be a clique constraint. Suppose, now, that there exists no other k such that $\pi_{ik} > 0$, i.e. $\pi_{ik} = 0$ for all $k \neq j$. Then, by defining a new vector $(\bar{x},\bar{y})$ to be $\bar{x}_{ij} = 0$ for all $j \in D$, $\bar{y}_i = 1$, others being same with (x,y), we get $(\bar{x},\bar{y}) \in L_I^{pd}$ and $\pi\bar{x} + \mu\bar{y} = \pi x + \mu y + \mu_i > 1$. This is a contradiction.

3) Since (π,μ) is not a clique constraint, from 2) of theorem 4.2, $|I_j| \geq 2$. Suppose, now, that the statement is wrong. In other words, assume $\pi_{ij} > \pi_{qj}$ for all q in P different from i. Since $\pi x + \mu y \leq 1$ is not a clique constraint, there exists an integer vector $(x,y) \in H_{\pi,\mu}$ such that $x_{ij} = 0$ and $y_i = 0$ for some $i \in I_j$, $j \in D$. Otherwise, we will have $x_{ij} + y_i = 1$ which would be a clique constraint. Now, if we define a new vector $(\bar{x},\bar{y})$ such that $\bar{x}_{qj} = 0$ for $q \neq i$, $\bar{x}_{ij} = 1$ and all other components equal to the components of (x,y), then we get $(\bar{x},\bar{y}) \in L_I^{pd}$ and $\pi\bar{x} + \mu\bar{y} = \pi x + \mu y + \pi_{ij} - \pi_{qj} > 1$. This is a contradiction.

4) Suppose $I_j = I_k \neq \emptyset$. First, we note that if $(x,y) \in H_{\pi,\mu}$ and $x_{ij} = 0$ for all $i \in I_j$, then we must have $y_i = 1$ for all $i \in I_j$. This is so because, if there exists $k \in I_j$ such that $y_k = 0$, then by defining a new vector $(\bar{x},\bar{y})$ to be $\bar{x}_{kj} = 1$, and $\bar{x}_{qj} = 0$ if $x_{qj} = 1$ (note: $q \notin I_j$), and all others being the same as (x,y), we get $(\bar{x},\bar{y}) \in L_I^{pd}$ and $\pi\bar{x} + \mu\bar{y} = \pi x + \mu y + \pi_{kj} > 1$, which leads to a contradiction. Thus, if $x_{ij} = 0$ for all $i \in I_j$, then $x_{ik} = 0$ for all $i \in I_k$ because $y_i = 1$ for all $i \in I_j$ and $I_j = I_k$. In other words, if $x_{ij} = 1$ for some $i \in I_j$, then there must

exist $i' \in I_k$ (possibly, $i = i'$) such that $x_{i'j} = 1$. Therefore, we have $\sum_{i \in I_j} x_{ij} = \sum_{i \in I_k} x_{ik}$ for $(x,y) \in H_{\pi,\mu}$ implying that we can not have $p(d+1)$ linearly independent vectors satisfying $\pi x + \mu y = 1$. This is a contradiction. Δ

Remark 5.1. Theorem 4.2 and theorem 5.1 together imply that, if $(\pi,\mu) \in F^{pd} - F_t^{pd}$, then either $\pi^j = 0$ or if $\pi^j \neq 0$ then at least two components π_{ij} and π_{qj}, say, of π^j are positive. Furthermore, for at least one $j \in D$ we have $\pi^j \neq 0$. If $\pi^j \neq 0$, then by part 2) of theorem 5.1 it follows that there exists a $k \neq j$ such that $I_k \neq \emptyset$, $\mu_i > 0$ where $i \in I_k$ and $i \in I_j$. Also, by part 4) of theorem 5.1, we have $I_k \neq I_j$. Finally, for each j such that $\pi^j \neq 0$ there exists an i such that $\mu_i > 0$ and $\pi_{ij} = 0$.

Remark 5.2. Theorem 4.2 and theorem 5.1 together imply that, if $(\pi.\mu) \in F^{pd}-F_t^{pd}$, then there must exist at least 3 plants i such that $\mu_i > 0$ and at least 3 destinations j such that $\pi^j \neq 0$. This is so because, if $\pi^j \neq 0$, then we must have $|I_j| \geq 2$ and at least one plant i such that $\pi_{ij} = 0$. Thus, we must have at least 3 plants i such that $\mu_i > 0$. Furthermore, if $i \in I_j$, then there must exist a destination q different from j such that $\pi_{iq} > 0$ in order to satisfy $|J_i| \geq 2$. Now, if $\pi^t = 0$ for $t \neq j,q$, then , for each i such that $\mu_i > 0$, we have $\pi_{ij} > 0$ if and only if $\pi_{iq} > 0$ in order to satisfy $|J_i| \geq 2$. This is a contradiction to the fact that $I_j \neq I_q$. Thus, there must exist at least 3 destinations j such that $\pi^j \neq 0$.

6. Facets for the case of 3 plants and several destinations ($d \geq 3$).

The results of the previous section enable us now to describe all facets for the case of three plants and several destinations.

Theorem 6.1. *Let $(\pi,\mu) \in F^{pd}-F_t^{pd}$ where $p = 3$ and $d \geq 3$. Then $\pi x + \mu v \leq 1$ is a facet if and only if $\mu_i = \frac{1}{4}$ for all $i \in P = \{1,2,3\}$ and $\pi_{ij} = \frac{1}{4}$ for $i \in I_j$, $j \in J$ where J is any subset of D consisting of exactly 3 elements and I_j's are distinct subsets of I with $|I_j| = 2$.*

Proof:

($\Leftarrow$)Without loss of generality, assume $J = \{1,2,3\}$, $I_1 = \{1,2\}$, $I_2 = \{2,3\}$ and $I_3 = \{1,3\}$. Then, we have to show $\sum_{j\in J} \sum_{i\in I_j} \frac{1}{4} x_{ij} + \sum_{i\in P} \frac{1}{4} y_i \leq 1$ is a facet for L_I^{pd}. This follows from Guignard [16].

($\Rightarrow$)By remark 5.1, we have $I_j = \emptyset$ or $|I_j| = 2$ for all $j = 1,2,\ldots,d$ and $|I_j| = 2$ for at least one j. Furthermore, it follows from remark 5.1 that $\mu_i > 0$ for $i = 1,2,3$.

From 2) of theorem 5.1, we know that $|J_i| \geq 2$ for all $i = 1,2,3$. Suppose $|J_i| = 3$, say, $J_1 = \{1,2,3\}$. Then, in order to satisfy $|I_1| = 2$, we must have either $I_1 = \{1,2\}$ or $I_1 = \{1,3\}$. If $I_1 = \{1,2\}$, then I_2 must be $\{1,3\}$ in order to satisfy $I_1 \neq I_2$. But, this would force I_3 to be equal to either I_1 or I_2 contradicting the fact that $\pi x + \mu y \leq 1$ is a facet. Similar result holds if we start with $I_1 = \{1,3\}$. Also, it can be easily seen that if $|J_i| > 3$, then we would get a similar contradiction. Therefore, $|J_i| = 2$.

Now, since $|J_i| = 2$ for all $i = 1,2,3$ and I_j' s are distinct with $|I_j| = 2$ for all $j = 1,2,\ldots,|J|$, it is obvious that $|J| = 3$ where $J = \{j \in D \mid \pi^j \neq 0\}$. Note that I_j's must be distinct subsets of I with $|I_j| = 2$, and thus I_j for $j = 1,2,3$ is one element of $\{\{1,2\},\{1,3\},\{2,3\}\}$.

Next, we show that all of the π_{ij}'s and μ_i's have to be same. Without loss of generality, assume $I_1 = \{1,2\}$, $I_2 = \{2,3\}$ and $I_3 = \{1,3\}$. We note that there exists a solution vector $(x,y) \in H_{\pi,\mu}$ such that $x_{11} = 0$ and $y_1 = 0$. This, in turn, implies that $x_{13} = 1$. (Otherwise, if $x_{13} = 0$, then letting $y_1 = 1$ would contradict the fact that $\pi x + \mu y \leq 1$ is a facet.) From theorem 5.1, we know that $\mu_1 \geq \pi_{13}$. Suppose $\mu_1 > \pi_{13}$. Then, letting $y_1 = 1$ and $x_{13} = 0$ would contradict the fact that $\pi x + \mu y \leq 1$ is a facet. Therefore, $\mu_1 = \pi_{13}$. Again, from theorem 5.1, $\pi_{13} = \pi_{33}$. Repeating this argument, one can show that all the coefficients are same and so dividing by π_{11}, $\pi x + \mu y \leq 1$ becomes $\sum_{j \in J} \sum_{i \in I_j} x_{ij} + \sum_{i \in P} y_i \leq \frac{1}{\pi_{11}}$. However, the left hand side of the above inequality corresponds to a chordless odd cycle of length 9 and thus the right hand side is equal to 4. Therefore, $\pi_{11} = \frac{1}{4}$.

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